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# The charge quantisation condition for Dirac dyons

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**Abstract.** Using a fibre bundle approach as in the work of Wu and Yang, we find that the charge quantisation condition for two dyons each with Dirac magnetic charge $g_i$ and charge $q_i$ is $q_1 g_2 = \frac{1}{2}\hbar c n_1$ and $q_2 g_1 = \frac{1}{2}\hbar c n_2$, where $n_1$ and $n_2$ are integers. This is more restrictive than the condition $q_1 g_2 - q_2 g_1 = \hbar c n$ found by Schwinger and Zwanziger using an infinite string of singularities. Our condition does not allow a new elementary unit of electric charge for dyons contrary to their condition. Our quantisation condition agrees with the result one would obtain using a minimal, semi-infinite string of singularities, just as the result of Wu and Yang agrees with the Dirac version of the quantisation condition but not the Schwinger version for the case of an uncharged magnetic monopole.

## 1. Introduction

Dirac (1931, 1948) first considered the possibility of a magnetic monopole and found that the magnetic charge $g$ was related to the charge on the electron by $eg = \frac{1}{2}n\hbar c$ where $n$ is an integer. He formulated his theory in terms of a minimal semi-infinite string of singularities in the vector potential, $A_\mu$, terminating on the magnetic charge. Schwinger (1966) used an infinite string of singularities with the magnetic charge in the centre and found $eg = n\hbar c$ instead (see also Peres 1968). Wu and Yang (1975) discussed magnetic monopoles in the much more elegant language of fibre bundles. Here a magnetic monopole is a non-trivial principal fibre bundle with structure group $U_1$ over base space $R_3 - \{\phi\}$ which can be contracted to $S_2$. Strings of singularities for the magnetic monopoles are replaced by the fact that at least two different open sets are now required to cover the base space with the vector potential (connection in the fibre bundle language) defined differently in the two different regions. They found

$$A_{\phi_a} = \frac{g}{r \sin\theta}(1 - \cos\theta) \tag{1}$$

$$A_{\phi_b} = \frac{-g}{r \sin\theta}(1 + \cos\theta) \tag{2}$$

where region a is $\{0 \leq \theta < \pi/2 + \delta, 0 < r, 0 \leq \phi < 2\pi, \text{all } t\}$ and region b is $\{\pi/2 - \delta < \theta \leq \pi, 0 < r, 0 \leq \phi < 2\pi, \text{all } t\}$. These two vector potentials are related by a gauge transformation (transition function in the fibre bundle language) in the overlap region near the equator of $S_2$. Requiring the phase factor of an electron wavefunction to be single-valued under this gauge transformation in the overlap region leads to the quantisation condition $eg = \frac{1}{2}n\hbar c$ in agreement with Dirac but not Schwinger.

In this paper we will look at the quantisation condition for dyons, particles that contain both electric and magnetic charges. Schwinger (1968, 1969) considered such particles and found for two dyons with charges $(q_1, g_1)$ and $(q_2, g_2)$ that the quantisation condition becomes

$$q_1 g_2 - q_2 g_1 = \hbar c n. \tag{3}$$

Using an infinite antisymmetric string function, Zwanziger (1968) found this same result and pointed out that it allows dyons to have an elementary unit of electric charge unrelated to the charge on the electron. (Other authors (Witten 1979) have used (3) but with an additional factor of $\frac{1}{2}$ inserted on the right-hand side. This latter version can be derived heuristically by considering the angular momentum in the field of the two-dyon system (Saha 1936, Fierz 1944, Wilson 1949).) The combination of charges on the left-hand side of (3) is invariant under a duality rotation (like a chiral transformation in quantum theory), as are Maxwell's equations with electric and magnetic sources present. If the semi-infinite string function is used in Zwanziger's (1968) work, instead of (3), we find

$$q_1 g_2 = \tfrac{1}{2} n_1 \hbar c \qquad q_2 g_1 = \tfrac{1}{2} n_2 \hbar c \tag{4}$$

as we might expect from the usual Dirac quantisation condition. This does not allow dyons to have a new elementary unit of electric charge.

In this paper we extend the work of Wu and Yang (1975) to two (or more) dyons. Even though this approach is based on the fibre bundle description of gauge theories, we will avoid the fibre bundle language for the most part. Fundamentally, however, it is important to realise that a magnetic monopole is a non-trivial principal fibre bundle and to treat it as such. This avoids the uncertainties of not knowing what type of string singularities to use and becomes important when different string versions make very different physical predictions as in (3) and (4) above. We will show that a generalised Wu and Yang approach gives (4) above and not (3) in agreement with a (minimal) semi-infinite string. Thus a new elementary unit of electric charge on dyons appears ruled out. Let us look at this in detail now.

## 2. Dyon quantisation condition

Consider two dyons with electric and magnetic charges $(q_1, g_1)$ and $(q_2, g_2)$ respectively. We wish to extend the work of Wu and Yang (1975) to this case. Now a minimum of three overlapping open sets are required. Let the two dyons be on the $z$ axis with $(q_1, g_1)$ at $z = +d$ and $(q_2, g_2)$ at $z = -d$ in a cylindrical coordinate system. We can choose the three open sets as follows:

$$\begin{aligned} &\text{set A: } z > d && 0 \leqslant \phi < 2\pi && r > 0 \\ &\text{set B: } z < r^2 + d^2 && 0 \leqslant \phi < 2\pi && r > 0 \\ &\text{set C: } z < -d && 0 \leqslant \phi < 2\pi && r > 0. \end{aligned} \tag{5}$$

Note that set B overlaps both A and C but that A and C do not overlap each other. Set B is the region between the two sheets of the hyperboloid $z^2/d^2 - r^2/d^2 = 1$ which crosses the $z$ axis at points $z = \pm d$. Set B extends upward in the positive $z$ direction into set A and downward in the negative $z$ direction into set B. Other choices of these sets are of course possible.

The electromagnetic vector potential describing the magnetic monopoles (but not the electric charges) is then given in the three regions by

$$
\begin{aligned}
A_{\phi_A} &= \frac{-g_1(z-d)}{r[r^2+(z-d)^2]^{1/2}} + \frac{g_1}{r} - \frac{g_2(z+d)}{r[r^2+(z+d)^2]^{1/2}} + \frac{g_2}{r} \\
A_{\phi_B} &= \frac{-g_1(z-d)}{r[r^2+(z-d)^2]^{1/2}} - \frac{g_1}{r} - \frac{g_2(z+d)}{r[r^2+(z+d)^2]^{1/2}} + \frac{g_2}{r} \\
A_{\phi_C} &= \frac{-g_1(z-d)}{r[r^2+(z-d)^2]^{1/2}} - \frac{g_1}{r} - \frac{g_2(z+d)}{r[r^2+(z+d)^2]^{1/2}} - \frac{g_2}{r}.
\end{aligned}
\tag{6}
$$

These vector potentials are finite everywhere in their respective regions except at the position of the charges, assuming the positive square root is always taken so that, for example, $[r^2+(z-d)^2]^{1/2} \to |z-d|$ as $r \to 0$. As $r \to 0$ they all approach zero. One can easily show that taking the curl of the vector potential gives the correct magnetic field everywhere. No infinite or semi-infinite strings of singularities appear. The electric charges on the dyons can also be included through the usual $\mu = 0$ component of the electromagnetic vector potential $A_\mu$. This is

$$
A_0 = \frac{q_1}{[r^2+(z-d)^2]^{1/2}} + \frac{q_2}{[r^2+(z+d)^2]^{1/2}}
\tag{7}
$$

in all three regions A, B and C. It is also finite everywhere except the position of the dyons. (6) and (7) together give $A_\mu$ everywhere.

Now we are interested in the regions where B overlaps A and where B overlaps C. In these overlap regions we have

$$
\begin{aligned}
A_{\phi_B} - A_{\phi_A} &= -2g_1/r \\
A_{\phi_C} - A_{\phi_B} &= -2g_2/r.
\end{aligned}
\tag{8}
$$

These two vector potentials must be related by a gauge transformation. We know that particles carrying electric charge $e$ but not magnetic charge exist. Consider now an electron moving near the two dyons. Under a gauge transformation of the form

$$
A_{\mu_B} = A_{\mu_A} + \frac{\hbar c}{e}\frac{\partial \alpha_{AB}}{\partial x^\mu}
\tag{9}
$$

the electron wavefunction will change according to

$$
\psi_B = \mathrm{e}^{\mathrm{i}\alpha_{AB}}\psi_A
\tag{10}
$$

since the electron wavefunction in general has a phase factor (Wu and Yang 1975) $\exp(\mathrm{i}e/\hbar c)\int_P^Q A_\mu \, dx^\mu$ for any path from P to Q. This phase factor gives the usual minimally coupled gauge covariant $\partial_\mu + (\mathrm{i}e/\hbar c)A_\mu$ derivative, for an electron interacting with an external electromagnetic field (Mandelstam 1962). (8) and (9) then give

$$
\begin{aligned}
\alpha_{AB} &= -2eg_1\phi/\hbar c \\
\alpha_{BC} &= -2eg_2\phi/\hbar c.
\end{aligned}
\tag{11}
$$

Requiring the electron wavefunction to be single-valued then gives from (10) and (11) that

$$
eg_1 = \tfrac{1}{2}n_1\hbar c \qquad eg_2 = \tfrac{1}{2}n_2\hbar c.
\tag{12}
$$

This is very similar to the work of Wu and Yang and it is not surprising that the magnetic charges on the dyons are quantised according to the usual Dirac quantisation condition.

Having looked at electron wavefunctions, let us now consider the interaction of one dyon with the other and look at the wavefunctions of the dyons themselves. These also must be single-valued under gauge transformations. If we consider dyon 1 with charges $(q_1, g_1)$ interacting with an external electromagnetic field, we have that $q_1$ appears in gauge-covariant derivatives as $\partial_\mu + (\mathrm{i}q_1/\hbar c)A_\mu$ where $A_\mu$ are electromagnetic fields other than those generated by $(q_1, g_1)$. A convenient way to write the interaction of the magnetic monopole with the external field is in terms of an additional term $\mathrm{i}g_1 B_\mu/\hbar c$ in the above gauge-covariant derivative. The dyon wavefunction in general then has a phase factor

$$\exp\left(\frac{\mathrm{i}q_1}{\hbar c}\int_{\mathrm{P}}^{\mathrm{Q}} A_\mu \,\mathrm{d}x^\mu + \frac{\mathrm{i}g_1}{\hbar c}\int_{\mathrm{P}}^{\mathrm{Q}} B_\mu \,\mathrm{d}x^\mu\right). \tag{13}$$

This $B_\mu$ is related to $A_\mu$ by

$$B_{\alpha|\beta} - B_{\beta|\alpha} = \tfrac{1}{2}\varepsilon_{\alpha\beta}{}^{\gamma\delta}(A_{\gamma|\delta} - A_{\delta|\gamma}). \tag{14}$$

$B_\mu$ is the potential for the dual of the electromagnetic field tensor with

$$\tilde{F}_{\mu\nu} \equiv B_{\mu|\nu} - B_{\nu|\mu}. \tag{15}$$

Note that $A_{\gamma|\delta} - A_{\delta|\gamma}$ is well defined on the right-hand side of (14) even when $A_\mu$ is defined differently on different open sets, since this is just the continuous electromagnetic field tensor $F_{\gamma\delta}$.

Now if dyon 1 above is interacting with a second dyon with charges $(q_2, g_2)$, this second dyon will act as the source of the $A_\mu$ and $B_\mu$ in (13). When looking at the interaction of $q_1$ with an external field it is most convenient to work in terms of $A_\mu$ in (13). $B_\mu$ is more convenient when looking at the interactions of the magnetic monopole charge $g_1$ in (13). External fields must be expressed *either* in terms of $A_\mu$ or in terms of $B_\mu$ but we cannot use *both* $A_\mu$ and $B_\mu$ at the same time in describing these sources. (Our use of $B_\mu$ here differs significantly from the way in which it is used in early work on magnetic monopoles by Cabibbo and Ferrari (1962). In their work $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + \varepsilon^{\mu\nu\rho\sigma}\partial_\rho B_\sigma$ and a single set of potentials is used for all regions. Their derivatives do not obey the Jacobi identities.)

We need the $A_\mu$ produced by dyon 2. We will need two regions to describe the magnetic monopole $g_2$. Following Wu and Yang we can use their regions and $A_{\phi_a}$ and $A_{\phi_b}$ in (1) and (2) above with $g \to g_2$. The electric charge on dyon 2 will also give rise to an $A_0$ which we can write as

$$A_0 = q_2/r \tag{16}$$

in both regions in spherical coordinates. We also need the $B_\mu$ produced by dyon 2. Note that, since $B_\mu$ is the potential for the dual of the electromagnetic field tensor, duality invariance tells us that the roles of the electric and magnetic monopoles are reversed now. That is, a magnetic monopole produces a trivial $B_\mu$ which can be defined as

$$B_0 = g_2/r \tag{17}$$

in both of Wu and Yang's regions, whereas an electric monopole now requires more than one open set for its description. We have for the electric charge that

$$B_{\phi_a} = \frac{q_2}{r \sin\theta}(1-\cos\theta) \tag{18}$$

and

$$B_{\phi_b} = -\frac{q_2}{r \sin\theta}(1+\cos\theta) \tag{19}$$

in their two regions a and b.

Now the wavefunction of dyon 1 with phase factor (13) must be single-valued under the gauge transformation which takes $A_{\mu_b}$ and $A_{\mu_a}$. Under the gauge transformation

$$A_{\mu_b} = A_{\mu_a} + \frac{\hbar c}{q_1}\frac{\partial \alpha_{ab}}{\partial x^\mu} \tag{20}$$

we have that the dyon wavefunction will change according to

$$\psi_b = e^{i\alpha_{ab}}\psi_a \tag{21}$$

where

$$\alpha_{ab} = -\frac{2q_1 g_2}{\hbar c}\phi. \tag{22}$$

Requiring the dyon wavefunction to be single-valued then gives

$$q_1 g_2 = \tfrac{1}{2}n_1 \hbar c. \tag{23}$$

The wavefunction of dyon 1 with phase factor (13) must also be single-valued under the gauge transformation which takes $B_{\mu_b}$ into $B_{\mu_a}$. When looking at the interaction of $g_1$ with the other dyon, it is most convenient to work in terms of $B_\mu$. $B_{\mu_a}$ and $B_{\mu_b}$ are given in (17)-(19). The dyon 1 wavefunction will change according to

$$\psi_b = e^{i\beta_{ab}}\psi_a \tag{24}$$

under

$$B_{\mu_b} = B_{\mu_a} + \frac{\hbar c}{g_1}\frac{\partial \beta_{ab}}{\partial x^\mu}. \tag{25}$$

Using (17)-(19) gives

$$\beta_{ab} = -\frac{2q_2 g_1 \phi}{\hbar c}. \tag{26}$$

Requiring the dyon 1 wavefunction to be single-valued under this gauge transformation then gives the quantisation condition

$$g_1 q_2 = \tfrac{1}{2}n_2 \hbar c. \tag{27}$$

Note that the gauge transformations (20) and (25) are completely independent because of the structure of (14). The dyon 1 wavefunction must be single-valued under (20) and (25) independently. Thus we obtain both (23) and (27). These are our dyon quantisation conditions.

## 3. Discussion

Our dyon quantisation conditions (23) and (27) agree with the result (4) Zwanziger (1968) obtains if he uses a semi-infinite string function. It is interesting that the Wu–Yang (1975) fibre bundle approach followed here leads to the quantisation condition for both ordinary magnetic monopoles and for dyons that one would obtain using a minimal semi-infinite string of singularities rather than an infinite string of singularities. Our conditions are consistent with (3), but more restrictive. (12), (23) and (27) taken together imply that dyons containing Dirac (1931, 1948) type magnetic monopoles must have electric charges which are integer multiples of the charge on the electron. The Schwinger (1968, 1969) and Zwanziger (1968) condition (3) would allow dyons to carry a second elementary unit of electric charge unrelated to the charge on the electron (Zwanziger 1968). For completeness we should mention the interesting work by Witten (1979) who shows that in a theory with a *CP* violating term $\theta e^2(32\pi^2)^{-1}F_{\mu\nu}\tilde{F}_{\mu\nu}$, a dyon can have a charge $ne - e\theta/2\pi$. The dyons he discusses involve 't Hooft (1974)–Polyakov (1974) magnetic monopoles (early work on Yang–Mills monopoles was also done by Joseph (1972) and Kerner (1975)) rather than the Dirac type monopoles discussed in this paper and above. The two different types are quite different objects mathematically (Ezawa and Tze 1976). Nonetheless, since there is no measurable distinction between Abelian and non-Abelian poles if one is far away, our results also can be applied to non-Abelian poles.

Schwinger (1968, 1969) and Zwanziger (1968) seem to have been led to the quantisation condition (3) on $q_1g_2 - q_2g_1$ primarily because this combination is duality rotation invariant. Since particles like the electron exist, giving quantisation conditions like (12) which are clearly not duality rotation invariant, there seems to be no good physical reason to insist that a dyon be treated in a duality invariant fashion.

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